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AN ARBITRAGE DRIVEN PRICE DYNAMICS OF AUTOMATED MARKET MAKERS IN THE PRESENCE OF FEES

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Abstract. Under a reference market price following a geometric Brownian motion, we present a model for price dynamics in the Automated Market Makers (AMM) setting. The AMM price is constrained within bounds determined by constant multiples of this reference price. By employing local time and excursion theory, we derive several analytical results, including a time-changed representation of the AMM price process and its asymptotic behavior.

1. **Introduction.** Automated Market Makers (AMMs) [2, 5] represent a significant innovation in decentralized finance, leveraging blockchain technology to automate pricing and order matching on decentralized exchanges. Unlike traditional centralized limit order book models, AMMs determine prices based on the available liquidity in pools funded by liquidity providers (LPs). This approach enables secure, peer-to-peer trading of crypto assets without intermediaries, making the process accessible to anyone with a self-custody wallet and compatible tokens.

A critical challenge for LPs in AMMs is adverse selection, primarily due to arbitrageurs exploiting price discrepancies between AMMs and real-time market prices on centralized exchanges. Recent studies, such as [7, 8], have quantified LPs' losses due to arbitrage using the 'loss-versus-rebalancing' (LVR) metric.

In contrast to traditional financial markets dominated by large institutional players, AMMs democratize access to market-making activities. This democratization

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can lead to a more distributed and resilient financial ecosystem, promoting financial inclusion and reducing systemic risks associated with centralized market failures.

Trading fees in AMMs introduce economic considerations that influence market participants' behavior. These fees deter excessive arbitrage activity, which, while necessary for market efficiency, can lead to significant losses for liquidity providers. Optimizing fee structures is crucial for balancing sufficient liquidity with minimal adverse impacts on LPs, ensuring the long-term viability of decentralized exchanges.

Despite its importance, the existing results on the mathematical analysis of the AMM price process are limited. In particular, advanced stochastic process theory has yet to be introduced in this field of research. This paper proposes an AMM pricing model based on three key assumptions:

(a) A reference market with infinite liquidity and no trading costs exists, where the reference market price $p = (p_t)_{t \geq 0}$ follows a geometric Brownian motion.
(b) The AMM applies a fixed trading fee tier of $(1 - \gamma)\%$, proportional to the trading volume. We focus on the worst-case scenario, assuming all trades within the pool are arbitrage-driven.
(c) Arbitrageurs actively monitor the market, initiating trades whenever arbitrage opportunities arise.

These assumptions combine elements from [8] (continuous arbitrage without fees) and [7] (discrete arbitrage with fees). Our model addresses continuous arbitrage while incorporating fees.

From these assumptions, we derive:

- Arbitrage opportunities exist when $p_t < \gamma \tilde{p}_t$ or $p_t > \frac{1}{\gamma}\tilde{p}_t$ (see [1, §2.4] and [4, §3]).
- The AMM price $\tilde{p}_t$ remains stable within the no-arbitrage interval of $\gamma p_t \leq \tilde{p}_t \leq \frac{1}{\gamma} p_t$.
- Arbitrage actions occur only at the boundaries of the no-arbitrage interval, i.e., when $p_t = \gamma \tilde{p}_t$ or $p_t = \frac{1}{\gamma}\tilde{p}_t$.

Figure 1 illustrates a sample path of the logarithm of the AMM price $\tilde{p}$, the logarithm of the reference market price p, and the boundaries of the no-arbitrage interval. Notably, the AMM price is adjusted upwards (resp. downwards) when it reaches the lower (resp. upper) boundary and remains constant otherwise.

Our work builds upon insights from Tassy and White's unpublished paper [11], extending their model beyond Constant Product Market Makers to encompass all AMM mechanisms.

This paper's key contribution is a detailed probabilistic description of the AMM price process utilizing Brownian motion local time. We show that the logarithm of $\tilde{p}$ can be expressed as a concatenation of the market price's running infimum and supremum processes with appropriate shifts. Leveraging classical results on the Skorokhod equation for reflected Brownian motion, we derive analytical results for $\tilde{p}$, including its existence, uniqueness, time-changed representation, and asymptotic behavior.

While [6] offers an alternative description using a stochastic storage model, our approach based on local time provides a complementary perspective. Additionally, [3] shares similar arbitrage assumptions in a discrete-time setting and focuses on optimizing fee rates for liquidity providers.

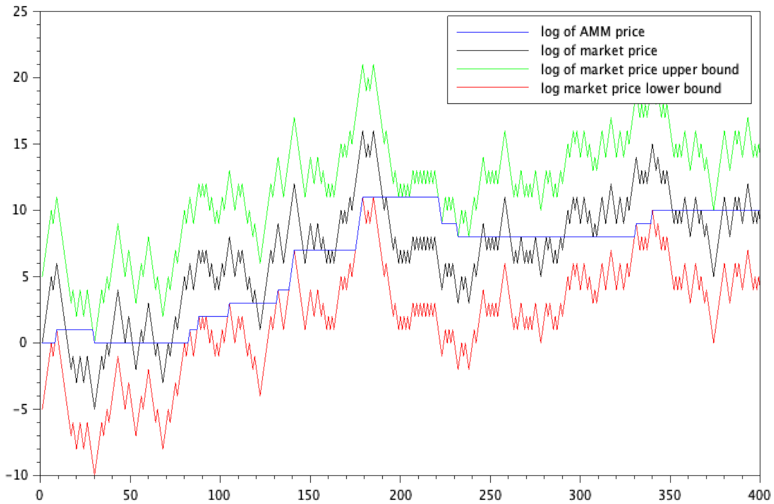

FIGURE 1. Figure 1. A sample path of the logarithm of the AMM price $\tilde{p}$ (blue) is shown, along with the logarithm of the reference market price p (black) and the boundaries of the no-arbitrage interval (green and red).

The remainder of this paper is organized as follows: We first present a precise mathematical construction of our AMM price process. Subsequently, we explore two distinct approaches to analyze the process under consideration, followed by discussing the economic implications and potential future research directions.

2. **AMM price process.** We assume the market price $p = (p_t)_{t\geq 0}$ (normalized in such a way that it is equal to 1 at time 0, and with the appropriate unit of time) is given by a geometric Brownian motion $p = \exp B$ for a Brownian motion with drift $B = (B_t)_{t\geq 0}$, and the AMM price $\tilde{p}$ is constrained by the inequalities

$$\gamma p \leq \tilde{p} \leq \gamma^{-1} p$$

where the parameter $\gamma \in (0, 1)$ is related to the fees of the AMM. These inequalities are equivalent to

$$B_t - c \leq U_t \leq B_t + c \tag{1}$$

where, for $t \geq 0$, U_t is $\log \tilde{p}$ at time t, B_t is $\log p$ at time t, and $c := \log(\gamma^{-1}) > 0$; see Figure 1.

The process $U = (U_t)_{t\geq 0}$ is chosen in such a way that $U_0 = B_0 = 0$, and U remains constant at any time where this is compatible with the inequalities (1). More precisely, it is constant except when it is pushed up by the process $(B_t - c)_{t\geq 0}$ or pushed down by $(B_t + c)_{t\geq 0}$ in order to preserve the inequalities (1). The main objective of this section is to present and verify Proposition 1, which defines the process U as a unique continuous process of bounded variation that satisfies additional monotonicity and boundedness conditions.

To this end, we first provide its recursive construction in terms of a sequence of stopping times. Specifically, we will repeatedly concatenate the running infimum of the process $B + c$ (horizontally shifted) and the running supremum of the process $B - c$ (horizontally shifted) in turn (see the blue process in Figure 1).

If B reaches c before $-c$ (we call this event A), then we define, by induction:

- T_0 as the infimum of $t \geq 0$ such that $B_t = c$.
- For all $k \geq 0$, T_{2k+1} is the first time $t \geq T_{2k}$ such that $B_t - \sup_{T_{2k} \leq s \leq t} B_s = -2c$.
- For all $k \geq 0$, T_{2k+2} is the first time $t \geq T_{2k+1}$ such that $B_t - \inf_{T_{2k+1} \leq s \leq t} B_s = 2c$.

By the well-known property of the reflected Brownian motion, $T_{m+1} - T_m$ is finite, and so is T_m almost surely (a.s.) for each $m \geq 0$. Then, we define $(U_t)_{t\geq 0}$ as follows:

- $U_t = 0$ for $0 \leq t \leq T_0$.
- For all $k \geq 0$ and $T_{2k} < t \leq T_{2k+1}$, $U_t = \sup_{T_{2k} \leq s \leq t} B_s - c$.
- For all $k \geq 0$ and $T_{2k+1} < t \leq T_{2k+2}$, $U_t = \inf_{T_{2k+1} \leq s \leq t} B_s + c$.

Informally, U is pushed up by $B - c$ in the intervals of time $[T_{2k}, T_{2k+1}]$ and pushed down by $B + c$ in the intervals of time $[T_{2k+1}, T_{2k+2}]$. Similarly, if B reaches $-c$ before c (event A^c), then we define:

- T_0 as the infimum of $t \geq 0$ such that $B_t = -c$.
- For all $k \geq 0$, T_{2k+1} is the first time $t \geq T_{2k}$ such that $B_t - \inf_{T_{2k} \leq s \leq t} B_s = 2c$.
- For all $k \geq 0$, T_{2k+2} is the first time $t \geq T_{2k+1}$ such that $B_t - \sup_{T_{2k+1} \leq s \leq t} B_s = -2c$.

Then,

- $U_t = 0$ for $0 \leq t \leq T_0$.
- For all $k \geq 0$ and $T_{2k} < t \leq T_{2k+1}$, $U_t = \inf_{T_{2k} \leq s \leq t} B_s + c$.
- For all $k \geq 0$ and $T_{2k+1} < t \leq T_{2k+2}$, $U_t = \sup_{T_{2k+1} \leq s \leq t} B_s - c$.

Proposition 1. *The process $U := (U_t)_{t\geq 0}$ is the only continuous process with finite variation satisfying the following properties*

(a) *$U_0 = 0$,*
(b) *$B_t - c \leq U_t \leq B_t + c$ for all $t \geq 0$,*
(c) *U is nondecreasing in all intervals where it remains strictly smaller than $B+c$,*
(d) *U is nonincreasing in all intervals where it remains strictly larger than $B - c$.*

Notice that the last two properties imply that U is constant on intervals where $B-c < U < B + c$.

Proof. We prove the proposition in three steps. First, we show the continuity of U and demonstrate that it has paths of bounded variation, which directly follows from the construction of U described above. Second, we prove properties (a)-(d). Finally, we establish the uniqueness of U using Skorokhod's lemma.

Proof for continuity and path variation. The process U is clearly continuous on $\mathbb{R}\backslash(T_m)_{m\geq 0}$. A careful look at the definition above shows that U has no discontinuity at times $(T_m)_{m\geq 0}$. Indeed, on A,

$$\begin{aligned}\lim_{t\uparrow T_{2k+1}} U_t &= \lim_{t\uparrow T_{2k+1}} \left(\sup_{T_{2k}\leq s\leq t} B_s - B_t\right) + \lim_{t\uparrow T_{2k+1}} B_t - c\\ &= 2c + B_{T_{2k+1}} - c\\ &= U_{T_{2k+1}},\\ \lim_{t\uparrow T_{2k+2}} U_t &= \lim_{t\uparrow T_{2k+2}} \left(\inf_{T_{2k+1}\leq s\leq t} B_s - B_t\right) + \lim_{t\uparrow T_{2k+2}} B_t + c\\ &= -2c + B_{T_{2k+2}} + c\\ &= U_{T_{2k+2}},\end{aligned} \tag{2}$$

where the second equalities hold because the supremum and infimum are attained at the hitting times T_{2k+1} and T_{2k+2} of reflected Brownian motion. The continuity holds in the same way on A^c as well.

Since U is monotone on each interval of the form $[T_m, T_{m+1}]$ for $m \geq 0$, it has finite variation.

Proof for properties (a)-(d). We now show properties (a)-(d) focusing on the case A. The proof for the other case is similar. We have

(i) U is equal to zero on $[0, T_0]$.

(ii) For all $k \geq 0$, U is nondecreasing in $[T_{2k}, T_{2k+1}]$ and nonincreasing in $[T_{2k+1}, T_{2k+2}]$.

(iii) For all $k \geq 0$, U is equal to $B - c$ at time T_{2k} and to $B + c$ at time T_{2k+1}.

Indeed, (i) holds by the definition of U and (iii) is clear from the definitions of the stopping times $(T_m)_{m\geq 0}$ (see also (2)). (ii) holds by the definition of U in terms of a piecewise supremum and infimum process, which is piecewise monotone. Now (a) holds by (i) and (b) holds by (ii) and (iii).

It remains to show (c) and (d); we only provide proof for (d) as that for (c) is similar. Let $J := \{t \geq 0 : U_t > B_t - c\}$. By (iii), $J \subset [0,\infty)\backslash(T_{2k})_{k\geq 0}$, which is the union of $[0, T_0)$ and $(T_{2k}, T_{2k+2}), k \geq 0$. We know from (i) that U is constant on $[0, T_0]$, and from (ii), that U is nonincreasing in $[T_{2k+1}, T_{2k+2})$. It is then enough to consider intervals

$$I \subset J \cap \left(\bigcup_{k\geq 0} (T_{2k}, T_{2k+1}]\right)$$

a sub-interval of $(T_{2k}, T_{2k+1}]_{k\geq 0}$ at which $U_t > B_t - c$. For $t \in I$, for some $k \geq 0$, $U_t = \sup_{T_{2k}\leq s\leq t} B_s - c$ and $U_t > B_t - c$, thus, the supremum of B on $[T_{2k}, t]$ is not reached at t. This implies that $\sup_{T_{2k}\leq s\leq t} B_s$ remains constant for $t \in I$, and thus U is constant, and then nonincreasing, in I, completing the proof of (d).

Proof for uniqueness. It remains to prove that the properties stated in Proposition 1 uniquely determine U.

Let U^1 and U^2 be two distinct processes satisfying the conditions (a)-(d), and let

$$S := \inf\{t \geq 0 : U^1_t \neq U^2_t\}. \tag{3}$$

We assume $S < \infty$ and derive contradiction.

Necessarily, by (a), (c) and (d), $U^1 = U^2 = 0$ until the first time where B reaches $\{-c, c\}$ which is strictly positive, and thus $S > 0$. By the definition of S as in (3), $U_t^1 = U_t^2$ for all $t \in [0, S)$, and by the continuity of U^1 and U^2, we also have $U_S^1 = U_S^2 =: U_S$.

Suppose

$$U_S^1 - B_S = U_S^2 - B_S = U_S - B_S \geq 0. \tag{4}$$

By this, continuity of B and U and (b), we have $B_t - c < U_t^j \leq B_t + c$ for $j = 1, 2$ on the interval $[S, S + u]$ for sufficiently small $u > 0$. For $j \in \{1, 2\}$, let us define the following processes on $[S, S + u]$:

$$\begin{aligned} A_t^j &:= U_S - U_t^j, \\ Y_t &:= B_t + c - U_S, \\ Z_t^j &:= Y_t + A_t^j = B_t + c - U_t^j. \end{aligned}$$

By (b), $Y_S \geq 0$ and $Z_t^j \geq 0$, and A_t^j is continuous and vanishes at the starting time S.

Moreover, the variation of A^j is carried by the set of times where $Z^j = 0$. Indeed, A^j changes only when U^j changes, or equivalently, only when U^j is equal to $B - c$ or $B + c$, and then only when $U^j = B + c$ since we know that $U^j > B - c$ on the interval $[S, S + u]$. From this last property and (d), U^j is nonincreasing in $[S, S + u]$, which implies that A^j is nondecreasing in this interval. By Skorokhod's lemma (see Lemma 2.1, Chapter VI of [9]; Lemma 3.1.2, Chapter 3 of [12]), A^j is uniquely determined by Y on the interval $[S, S + u]$, and thus $A^1 = A^2$, which implies that $U^1 = U^2$ on the interval $[S, S + u]$. This contradicts the fact that S is the infimum of times where $U^1 \neq U^2$ as in (3). This contradiction shows that $S = \infty$, i.e. one cannot have two distinct processes satisfying the assumptions of Proposition 1. Similar argument holds for the complement case of (4). □

Remark 1. As highlighted by Milionis et al. [7], discrete rebalancing times are crucial for accurately capturing adverse selection costs. Our core assumption is that arbitrage is performed continuously. While this simplifies the modeling, it presents notable limitations in practical scenarios. We aim to investigate the potential impact of block time and arbitrageur competition on adverse selection. Although our simplified model does not directly account for adverse selection costs, it intends to provide insights into how increased competition and reduced block time can influence the dynamics leading to such costs.

3. **Further analysis and limiting behavior.** In the sequel, we assume B_t is a standard Brownian motion. Let $W = (W_t)_{t \geq 0}$ be another standard Brownian motion. In our setting, the process $(U_t - B_t)_{t \geq 0}$, restricted to $[-c, c]$, behaves like a doubly reflected Brownian motion at $-c$ and c. In this section, we show that $(U_t - B_t)_{t \geq 0}$ and $(F(W_t))_{t \geq 0}$ have the same law, where F is a triangular function oscillating between $-c$ and c. Using the Itô-Tanaka formula (see Theorem 1.5 of [9]), we reconstruct B from W by integrating $F'(W_t)$ with respect to dW_t, and U as a combination of local times of W at levels where F' changes sign. See Chapter VI, Section 2 of [9] regarding local times.

Precisely, let $F : \mathbb{R} \to [-c, c]$ be the triangle wave function with period $4c$ such that

$$F(x) = \begin{cases} \cdots & \cdots, \\ -2c - x & \text{for } x \in [-3c, -c] \\ x & \text{for } x \in [-c, c], \\ 2c - x & \text{for } x \in [c, 3c], \\ \cdots & \cdots. \end{cases}$$

The derivative of F is 1 in the interval $((4k-1)c, (4k+1)c)$ and -1 in the interval $((4k+1)c, 4k+3)c)$, for all $k \in \mathbb{Z}$. The second derivative of F in the sense of the distribution is

$$2 \sum_{k \in \mathbb{Z}} (\delta_{(4k-1)c} - \delta_{(4k+1)c})$$

where δ_x denotes the Dirac measure at x. By the Itô-Tanaka formula,

$$F(W_t) = \beta_t - V_t, \quad t \geq 0,$$

where $(\beta_t)_{t \geq 0}$ is a standard Brownian motion,

$$V_t := \sum_{k \in \mathbb{Z}} (L_t^{(4k+1)c} - L_t^{(4k-1)c}), \quad t \geq 0,$$

with L_t^x denoting the local time of W at time t and level $x \in \mathbb{R}$.

Remark 2. The processes $(\beta_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ are continuous and enjoy the following properties:

(a) $V_0 = 0$.
(b) $(V_t)_{t \geq 0}$ has paths of finite variation, since it is the difference of two nondecreasing processes, given by sums of local times at different levels.
(c) For all $t \geq 0$, $V_t - \beta_t = -F(W_t) \in [-c, c]$, so $\beta_t - c \leq V_t \leq \beta_t + c$.
(d) In an interval $K := \{t \geq 0 : V_t < \beta_t + c\} = [0, \infty) \setminus \{t : F(W_t) = -c\}$, W does not reach levels congruent to $-c$ modulo $4c$ (these levels corresponding to values of $x \in \mathbb{R}$ such that $F(x) = -c$). Local times at levels $(4k-1)c$ for $k \in \mathbb{Z}$ are constant on K, which implies, from the definition of V, that V is nondecreasing on K.
(e) In an interval $K' := \{t \geq 0 : V_t > \beta_t - c\} = [0, \infty) \setminus \{t : F(W_t) = c\}$, W does not reach levels congruent to c modulo $4c$. Local times at these levels are constant on K', which implies that V is nonincreasing on K'.

From Proposition 1, and the corresponding uniqueness property, the process $(V_t)_{t \geq 0}$ can be recovered from $(\beta_t)_{t \geq 0}$ in the same way as $(U_t)_{t \geq 0}$ has been constructed from $(B_t)_{t \geq 0}$. Since $(B_t)_{t \geq 0}$ and $(\beta_t)_{t \geq 0}$ are both standard Brownian motions, $(B_t, U_t)_{t \geq 0}$ has the same joint distribution as $(\beta_t, V_t)_{t \geq 0}$.

We provide two descriptions of the processes discussed here using different time changes, the inverse local times and the hitting times.

3.1. **Time change with inverse local times.** We define the following linear combination of local times:

$$\left(\mathcal{L}_t := \sum_{k \in \mathbb{Z}} L_t^{(2k+1)c} \right)_{t \geq 0},$$

and its inverse process $(\sigma_\ell)_{\ell \geq 0}$ given by

$$\sigma_\ell := \inf\{t \geq 0, \mathcal{L}_t > \ell\}, \quad \ell \geq 0.$$

Note that $\mathcal{L}_t \nearrow \infty$ as $t \to \infty$, thus $\sigma_\ell < \infty$ for all $\ell \geq 0$ a.s.

Since $\mathcal{L}$ is an additive functional of the Brownian motion W (see [9], Chapter X), by the strong Markov property, the time-changed Brownian motion $(W_{\sigma_\ell})_{\ell \geq 0}$ is a continuous-time Markov chain. Moreover, since the variation of $(\mathcal{L}_t)_{t \geq 0}$ is supported on the set of times where W is in $E := \{(2k+1)c, k \in \mathbb{Z}\}$, we have that the Markov chain $(W_{\sigma_\ell})_{\ell \geq 0}$ takes its value only in E. This process can only jump between consecutive values in E. Indeed, if $(W_{\sigma_\ell})_{\ell \geq 0}$ jumps between non-consecutive values x and y in E at some $\ell \geq 0$, then the path $(W_t)_{\sigma_{\ell-} \leq t \leq \sigma_\ell}$ starts at x, ends at y, without accumulating local time at levels in E strictly between x and y: this event occurs with probability zero. Similarly, the starting point W_{σ_0} can only be c or $-c$. Thus, the following results are immediate.

Proposition 2. *The process $(W_{\sigma_\ell})_{\ell \geq 0}$ satisfies the following properties:*

(a) *It starts at c with probability $1/2$, and at $-c$ with probability $1/2$.*
(b) *It jumps by $2c$ or $-2c$, the rate of all possible jumps being the same, again by symmetry.*

Now, let us compute the rate of jumps. By translation invariance and the strong Markov property, the probability of having no jump during an interval of length ℓ is equal, for a Brownian motion B, its local time L^B at level zero and its inverse local time $\tau_\ell := \inf\{t \geq 0 : L^B > \ell\}$, $\ell \geq 0$:

$$\begin{aligned}
&\mathbb{P}[\max\{|B_s|, s \leq \tau_\ell\} < 2c] \\
&= (\mathbb{P}[\max\{B_s, s \leq \tau_\ell\} < 2c])^2 \\
&\quad \text{since the negative and positive excursions of } B \text{ are independent} \\
&= \left(\mathbb{P}[L^B_{\inf\{t \geq 0, B_t = 2c\}} > \ell]\right)^2 \\
&= (\mathbb{P}[\mathrm{BESQ}_2(2c) > \ell])^2 \\
&\quad \text{by Ray-Knight theorem (Theorem 52.1 of [10])}
\end{aligned}$$

where BESQ_2 denotes a squared Bessel process of dimension 2. Hence, for a standard exponential variable $\mathbf{e}$,

$$\begin{aligned}
&\mathbb{P}[\max\{|B_s|, s \leq \tau_\ell\} < 2c] \\
&= (\mathbb{P}[2 \times \mathbf{e} \times 2c > \ell])^2 \\
&= \left(e^{-\ell/4c}\right)^2 = e^{-\ell/2c}.
\end{aligned}$$

Hence, the sum of the rates of jump by $2c$ and $-2c$ is equal to $1/2c$, and then each of the two possible jumps has a rate $1/4c$ by symmetry.

For $\ell \geq 0$, we have

$$V_{\sigma_\ell} = \sum_{k \in \mathbb{Z}} (L^{(4k+1)c}_{\sigma_\ell} - L^{(4k-1)c}_{\sigma_\ell})$$

and

$$\mathcal{L}_{\sigma_\ell} = \sum_{k \in \mathbb{Z}} (L^{(4k+1)c}_{\sigma_\ell} + L^{(4k-1)c}_{\sigma_\ell}).$$

In an interval of values of ℓ where W_{σ_ℓ} remains constant and equal to $(4k-1)c$ for a given $k \in \mathbb{Z}$, the variations of V_{σ_ℓ} are opposite to the variations of $\mathcal{L}_{\sigma_\ell} = \ell$, since the only local time which varies in the corresponding sums is the local time at level $(4k-1)c$. Similarly, in an interval of values of ℓ where W_{σ_ℓ} remains constant and equal to $(4k+1)c$ for a given $k \in \mathbb{Z}$, the variations of V_{σ_ℓ} are equal to the variations of ℓ. Hence, $(V_{\sigma_\ell})_{\ell \geq 0}$ is the difference of time spent by the Markov process $(W_{\sigma_\ell})_{\ell \geq 0}$ at levels congruent to c modulo $4c$, minus the time spent at levels congruent to $-c$ modulo $4c$. If we reduce $(W_{\sigma_\ell})_{\ell \geq 0}$ modulo $4c$ and divide it by c, we get a Markov chain, $(Y_\ell)_{\ell \geq 0}$, on the two-state space $\{-1, 1\}$, with its starting point uniform in this space, and rate of jumps equal to $1/2c$. Then, $(V_{\sigma_\ell})_{\ell \geq 0}$ is the difference of time spent at state 1 by this Markov process, minus the time spent at state -1, i.e. the integral of the Markov chain: $V_{\sigma_\ell} = \int_0^\ell (1_{\{Y_s=1\}} - 1_{\{Y_s=-1\}}) ds$.

3.2. **Time change with hitting times.** Let us define the sequence of stopping times $(H_k)_{k \geq 0}$, as follows:

- If W hits c before $-c$ (we call this event $\mathcal{A}$), $H_0 = H_1$ is the first hitting time of c, and for $k \geq 1$,
$$H_{k+1} = \inf\{t > H_k, |W_t - W_{H_k}| \geq 2c\}.$$
- If W hits $-c$ before c (i.e. $\mathcal{A}^c$), H_0 is the first hitting time of $-c$, and for $k \geq 0$,
$$H_{k+1} = \inf\{t > H_k, |W_t - W_{H_k}| \geq 2c\}.$$

Between H_0 and H_1, V varies as the opposite of the local time of W at $-c$; for $k \geq 1$, between H_{2k-1} and H_{2k}, V varies as the local time of W at $W_{H_{2k-1}}$; for $k \geq 1$, between H_{2k} and H_{2k+1}, it varies as the opposite of the local time of W at $W_{H_{2k}}$. We deduce, for all $m \geq 1$,

$$V_{H_m} = \sum_{j=0}^{m-1} (-1)^{j-1} \left(L_{H_{j+1}}^{W_{H_j}} - L_{H_j}^{W_{H_j}} \right),$$

where we recall L_t^x denoting the local time of W at time t and level x. Notice that the term $j = 0$ vanishes on $\mathcal{A}$, since $H_0 = H_1$ in this case.

All these increments of local time, except the first one if $H_0 = H_1$ in the event $\mathcal{A}$, are independent and have the distribution of the local time $L_{T_{2c}^*}^B$ of B at level zero, where B is a standard Brownian motion and

$$T_{2c}^* = \inf\{t > 0, |B_t| \geq 2c\}.$$

Now, we show that using this approach, one can compute this distribution via Ray-Knight Theorem: for $\ell \geq 0$, $L_{T_{2c}^*}^B > \ell$ if and only if no excursion of B before the inverse local time τ_ℓ reaches $2c$ or $-2c$, which implies

$$\begin{aligned}
&\mathbb{P}(L_{T_{2c}^*}^B > \ell) = \mathbb{P}(\text{no excursion of } B \text{ reaches } \pm 2c \text{ before } \tau_\ell) \\
&= (\mathbb{P}(\text{no excursion of } B \text{ reaches } 2c \text{ before } \tau_\ell))^2 \text{ by independence of excursions} \\
&= (\mathbb{P}(\inf\{t \geq 0, B_t = 2c\} > \tau_\ell))^2 \\
&= \left(\mathbb{P}(L_{\inf\{t \geq 0, B_t = 2c\}}^B > \ell)\right)^2 \\
&= (\mathbb{P}(4c\mathbf{e} > \ell))^2 \text{ by Ray-Knight Theorem} \\
&= e^{-\ell/2c}.
\end{aligned}$$

Hence, $L^B_{T^*_{2c}}$ has the same distribution as $2c\mathbf{e}$. We deduce the equality in distribution, for $m \geq 2$,

$$V_{H_m} = 2c\left(-X\mathbf{e}_0 + \sum_{j=1}^{m-1}(-1)^{j-1}\mathbf{e}_j\right),$$

where $(\mathbf{e}_j)_{j\geq 0}$ is a sequence of i.i.d. standard exponential variables, and X is an independent, Bernoulli random variable with parameter $1/2$: $X = 0$ if and only if W hits c before $-c$ (i.e. $\mathcal{A}$).

For $k \geq 1$, $V_{H_{2k+1}}$ is a sum of $-2cX\mathbf{e}_0$ and of k i.i.d. random variables with the same distribution as $2c(\mathbf{e}_1 - \mathbf{e}_2)$. These k variables are centered, with variance equal to $8c^2$, since $\mathbf{e}_1$ and $\mathbf{e}_2$ are independent with variance 1. We get the following central limit theorem:

Proposition 3. *When k tends to infinity,*

$$\frac{V_{H_{2k+1}}}{c\sqrt{8k}} \longrightarrow \mathcal{N}(0,1)$$

in distribution.

The increment $H_{2k+1} - H_1$ is itself the sum of $2k$ i.i.d. random variables with the same distribution as T^*_{2c}. By the law of large numbers, we have a.s.

$$\frac{H_{2k+1}}{2k} \underset{k\to\infty}{\longrightarrow} \mathbb{E}[T^*_{2c}].$$

Stopping the martingale $(B_t^2 - t)_{t\geq 0}$ at time T^*_{2c}, we deduce for all $t \geq 0$, by optional sampling, that

$$\mathbb{E}[B^2_{\min(t,T^*_{2c})}] = \mathbb{E}[\min(t, T^*_{2c})].$$

For $t \to \infty$, $B^2_{\min(t,T^*_{2c})} \leq 4c^2$ almost surely converges to $4c^2$, and $\min(t, T^*_{2c})$ converges to T^*_{2c} from below. Applying dominated convergence to the left-hand side and monotone convergence to the right-hand side, we deduce

$$4c^2 = \mathbb{E}[T^*_{2c}],$$

and then almost surely

$$\frac{H_{2k+1}}{2k} \underset{k\to\infty}{\longrightarrow} 4c^2. \tag{5}$$

We get

$$\frac{V_{H_{2k+1}}}{\sqrt{H_{2k+1}}} = \frac{V_{H_{2k+1}}}{c\sqrt{8k}} \cdot 2c\left(\frac{H_{2k+1}}{2k}\right)^{-1/2}$$

where the first factor converges to a standard normal distribution by Proposition 3 and the second factor converges almost surely to 1 by (5). By Slutsky's theorem, we get the following:

Proposition 4. *When k tends to infinity,*

$$\frac{V_{H_{2k+1}}}{\sqrt{H_{2k+1}}} \longrightarrow \mathcal{N}(0,1)$$

in distribution.

This result can be compared to the central limit theorem

$$\frac{V_t}{\sqrt{t}} \underset{t\to\infty}{\longrightarrow} \mathcal{N}(0,1),$$

which is a direct consequence of the inequality

$$\beta_t - c \leq V_t \leq \beta_t + c,$$

where $(\beta_t)_{t\geq 0}$ is a Brownian motion.

4. **Conclusion.** This study presents a novel model for Automated Market Maker (AMM) price dynamics that incorporates continuous arbitrage and fee structures. Our key contributions include a rigorous mathematical framework using local time and excursion theory to analyze AMM price processes, explicit consideration of trading fees providing a more realistic representation of AMM behavior, and analytical results on the existence, uniqueness, and asymptotic behavior of the AMM price process.

The economic implications of our findings are significant. Our model demonstrates how fee structures can maintain price stability and protect liquidity providers from excessive arbitrage-related losses. These insights can inform the design of more robust and efficient AMM systems, potentially improving market performance and reducing risks for participants. Future research directions include extending the model to multi-asset AMMs and dynamic fee structures, empirical validation against real-world AMM data, and exploring interactions between several different AMM price dynamics.

In conclusion, our work advances both the theoretical understanding of AMM mechanisms and offers practical guidelines for optimizing economic outcomes in decentralized finance (DeFi). This research contributes to developing more efficient, resilient, and inclusive decentralized financial systems by elucidating the relationship between arbitrage, fees, and market stability.

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